

CLASSIFICATION OF KÄHLER HOMOGENEOUS MANIFOLDS OF NON-COMPACT DIMENSION TWO

SEYED RUHALLAH AHMADI AND BRUCE GILLIGAN

ABSTRACT. Suppose G is a connected complex Lie group and H is a closed complex subgroup such that $X := G/H$ is Kähler and the codimension of the top non-vanishing homology group of X with coefficients in \mathbb{Z}_2 is less than or equal to two. We show that X is biholomorphic to a complex homogeneous manifold constructed using well-known “basic building blocks”, i.e., \mathbb{C} , \mathbb{C}^* , Cousin groups, and flag manifolds. For H discrete the classification was presented in the first author’s dissertation [4].

1. INTRODUCTION

In this paper we consider complex homogeneous manifolds of the form G/H , where G is a connected complex Lie group and H is a closed complex subgroup of G . The existence of complex analytic objects on such a G/H , like non-constant holomorphic functions, plurisubharmonic functions and analytic hypersurfaces, is related to when G/H could be Kähler. So the first question one might consider concerns the existence of Kähler structures and we restrict ourselves to that question here. The structure of compact Kähler homogeneous manifolds is now classical [28] and [13] and the structure in the case of G -invariant metrics is also known [15]. As a consequence, our investigations here concern non-compact complex homogeneous manifolds having a Kähler metric that is not necessarily G -invariant.

Some results are known under restrictions on the type of group G that is acting. The base of the holomorphic reduction of any complex solvmanifold is always Stein [24], where the proof uses some fundamental ideas in [27]. For G a solvable complex Lie group and G/H Kähler the fiber of the holomorphic reduction of G/H is a Cousin group, see [32] and the holomorphic reduction of a finite covering of G/H is a principal Cousin group bundle, see [19]. If G is semisimple, then G/H admits a Kähler structure if and only if H is algebraic [11]. For G reductive there is the characterization that G/H is Kähler if and only if $S \cdot H$ is closed in G and $S \cap H$ is an algebraic subgroup of S , a maximal semisimple subgroup of G , see [20, Theorem 5.1]. There is also a result if G is the direct product of its radical and a maximal semisimple subgroup under some additional assumptions on the isotropy subgroup and so on the structure of X [31].

One way to proceed is to impose some topological restraints on X . In [17] we classified Kähler homogeneous manifolds X having more than one end by showing X is either a product of a Cousin group and a flag manifold or X admits a homogeneous

Date: January 30, 2015.

1991 Mathematics Subject Classification. Primary 32M10; Secondary 32Q15.

Key words and phrases. homogeneous Kähler manifold, non-compact dimension two.

fibration as a \mathbb{C}^* -bundle over the product of a compact complex torus and a flag manifold. Now in the setting of proper actions of Lie groups Abels introduced the notion of non-compact dimension, see [2] and [3, §2]. We do not wish to assume that our Lie group actions are necessarily proper ones, so we take a dual approach and define the non-compact dimension d_X of a connected smooth manifold to be the codimension of its top non-vanishing homology group with coefficients in \mathbb{Z}_2 , see §2. Our goal in this paper is to classify Kähler homogeneous manifolds G/H with $d_{G/H} = 2$. All such spaces are holomorphic fiber bundles where the fibers and the bases of the bundles involved consist of Cousin groups, flag manifolds, \mathbb{C} , and \mathbb{C}^* . We now present the statement of our main result, where T denotes a compact complex torus, C a Cousin group, and Q a flag manifold.

Theorem 1.1 (Main Theorem). *Suppose $X := G/H$ is Kähler, where G is a connected complex Lie group and H is a closed complex subgroup of G . Then $d_X = 2$ if and only if X is one of the following:*

Case I: H discrete: *A finite covering of X is biholomorphic to a product $C \times A$, with C a Cousin group, A a Stein Abelian Lie group and $d_C + d_A = 2$.*

Case II: H is not discrete:

- (1) *Suppose $\mathcal{O}(X) = \mathbb{C}$ and let $G/H \rightarrow G/N$ be its normalizer fibration.*
 - (a) *X is a $(\mathbb{C}^*)^k$ -bundle over $C \times Q$ with $d_C + k = d_X = 2$.*
 - (b) *X is $T \times G/N$ with $\mathcal{O}(G/N) = \mathbb{C}$ and G/N fibers as a \mathbb{C} -bundle over a flag manifold; there are two subcases depending on whether S acts transitively on G/N or not.*
- (2) *Suppose $\mathcal{O}(X) \neq \mathbb{C}$ and let $G/H \rightarrow G/J$ be its holomorphic reduction.*
 - (a) *G/J is Stein with $d_{G/J} = 2$ and $X = T \times Q \times G/J$, where*
 - (i) $G/J = \mathbb{C}$.
 - (ii) G/J is the 2-dimensional affine quadric
 - (iii) G/J is the complement of a quadric curve in \mathbb{P}_2
 - (iv) $G/J = (\mathbb{C}^*)^2$
 - (b) *G/J is not Stein with $d_{G/J} = 2$; then G/J is a \mathbb{C}^* -bundle over an affine cone minus its vertex and $X = T \times G/J$.*
 - (c) *$d_{G/J} = 1$ with G/J an affine cone minus its vertex and $d_{J/H} = 1$*
 - (i) $\mathcal{O}(J/H) = \mathbb{C}$: *X is a \mathbb{C}^* -bundle over $T \times Y$, where Y is a flag manifold bundle over the affine cone minus its vertex.*
 - (ii) $\mathcal{O}(J/H) \neq \mathbb{C}$: *this case does not occur.*

The paper is organized as follows. In section two we gather a number of technical tools. In particular, we note that Proposition 2.10 deals with the setting where the fiber of the normalizer fibration is a Cousin group and its base is flag manifold. It is essential for Case II (1) (a) in the Main Theorem and can be used to simplify the proof when $d_X = 1$, see Remark 2.11. Section three is devoted to the case when the isotropy subgroup is discrete. Sections four and five deal with general isotropy and contain the proof of the main result when there are no non-constant holomorphic functions and when there are non-constant holomorphic functions, respectively. In the last section we present some examples.

This work was partially supported by an NSERC Discovery Grant. We thank Prof. A. T. Huckleberry for his comments that led to improvements in the work. We also thank the Arbeitsgruppe Transformationsgruppen at the Ruhr-Universität

Bochum for their kind hospitality during a sojourn at which time part of this paper was written.

2. TECHNICAL TOOLS

The purpose of this section is to gather together a number of definitions and basic tools that are subsequently needed.

2.1. Basic Notions.

Definition 2.1. A *Cousin group* is a complex Lie group G with $\mathcal{O}(G) = \mathbb{C}$. The terminology *toroidal group* is also found in the literature. Every Cousin group is Abelian and is the quotient of \mathbb{C}^n by a discrete subgroup having rank $n + k$ for some k with $1 \leq k \leq n$. For details, we refer the reader to [1].

Definition 2.2. A *flag manifold* (the terminology *homogeneous rational manifold* is also in common usage) is a homogeneous manifold of the form S/P , where S is a connected semisimple complex Lie group and P is a parabolic subgroup of S . One source concerning the structure of flag manifolds is [8, §3.1].

Definition 2.3. For X a connected (real) smooth manifold we define

$$d_X := \dim_{\mathbb{R}} X - \min\{ r \mid H_k(X, \mathbb{Z}_2) = 0 \quad \forall \quad k > r \},$$

i.e., d_X is the codimension of the top non-vanishing homology group of X with coefficients in \mathbb{Z}_2 . We call d_X the *non-compact dimension* of X .

Proposition 2.4. Suppose X is a connected Stein manifold. Then $\dim_{\mathbb{C}} X \leq d_X$.

Proof. For X Stein one has $H_k(X) = 0$ for all $k > \dim_{\mathbb{C}} X$ by [34]. \square

2.2. Fibration Methods. Throughout we make use of a number of fibrations that are now classical.

- (1) **Normalizer Fibration:** Given G/H let $N = N_G(H^0)$ be the normalizer in G of the connected component of the identity H^0 of H . Since H normalizes H^0 , we have $H \subset N$ and the *normalizer fibration* $G/H \rightarrow G/N$.
- (2) **Holomorphic Reduction:** Given G/H we set $J := \{ g \in G \mid f(gH) = f(eH) \text{ for all } f \in \mathcal{O}(G/H) \}$. Then J is a closed complex subgroup of G containing H and we call the fibration $p : G/H \rightarrow G/J$ the *holomorphic reduction* of G/H . By construction G/J is holomorphically separable and $\mathcal{O}(G/H) \cong p^*(\mathcal{O}(G/J))$.

Suppose a manifold X admits a locally trivial fiber bundle $X \xrightarrow{F} B$ with F and B connected smooth manifolds. One would then like to know how d_F and d_B are related to d_X whenever possible. The following result was proved in §2 in [7] using spectral sequences.

Lemma 2.5 (The Fibration Lemma). Suppose $X \xrightarrow{F} B$ is a locally trivial fiber bundle with X, F, B smooth manifolds. Then

- (1) if the bundle is orientable (e.g., if $\pi_1(B) = 0$), then $d_X = d_F + d_B$.
- (2) if B has the homotopy type of a q -dimensional CW complex, then $d_X \geq d_F + (\dim B - q)$.
- (3) if B is homotopy equivalent to a compact manifold, then $d_X \geq d_F + d_B$.

Remark 2.6. If B is homogeneous, then one knows that B is homotopy equivalent to a compact manifold if:

- (1) the isotropy subgroup of B has finitely many connected components [30]; e.g., in an algebraic setting,
- (2) if B is a solvmanifold [29]; indeed, every solvmanifold is a vector bundle over a compact solvmanifold [9].

2.3. Special case of a question of Akhiezer. Later we will need a result that is based on [7, Lemma 8]. Since that Lemma was stated in a way suitable for its particular application in [7], we reformulate it in a form suitable for the present context.

Lemma 2.7 (Lemma 8 [7]). *Let G be a connected, simply connected complex Lie group with Levi-Malcev decomposition $G = S \ltimes R$ with $\dim_{\mathbb{C}} R = 2$ and Γ a discrete subgroup of G such that $X = G/\Gamma$ is Kähler. Then Γ is contained in a subgroup of G of the form $A \ltimes R$, where A is a proper algebraic subgroup of S .*

This has the following consequence which we use later.

Theorem 2.8. *Suppose G is a connected, simply connected, complex Lie group with Levi-Malcev decomposition $G = S \ltimes R$ with $\dim_{\mathbb{C}} R = 2$. Let Γ be a discrete subgroup of G such that $X = G/\Gamma$ is Kähler, Γ is not contained in a proper parabolic subgroup of G and $\mathcal{O}(G/\Gamma) \simeq \mathbb{C}$. Then $S = \{e\}$, i.e., G is solvable.*

Proof. The radical orbits are closed, e.g., see [18]. By lemma 2.7 the subgroup Γ is contained in a proper subgroup of G of the form $A \ltimes R$, where A is an algebraic subgroup of S . Thus there are fibrations

$$G/\Gamma \longrightarrow G/R \cdot \Gamma \longrightarrow S/A,$$

where the base $G/R \cdot \Gamma = S/\Lambda$ with $\Lambda := S \cap R \cdot \Gamma$. If A is reductive, then S/A is Stein and we get non-constant holomorphic functions on X as pullbacks using the above fibrations. But this contradicts the assumption that $\mathcal{O}(X) \simeq \mathbb{C}$. If A is not reductive then [25, Theorem, §30.1] applies and A is contained in a proper parabolic subgroup of S . But this implies Γ is too, thus contradicting the assumption that this is not the case. \square

2.4. The algebraic setting revisited. Throughout we repeatedly use two results of Akhiezer concerning the invariant d_X in the setting where $X = G/H$ and G is a connected linear algebraic group over \mathbb{C} and H is an algebraic subgroup of G . For the convenience of the reader we now state these here.

Theorem 2.9 ($d = 1$ [5]; $d = 2$ [6]). *Suppose G is a connected linear algebraic group over \mathbb{C} and H is an algebraic subgroup of G and $X := G/H$.*

- (1) $d_X = 1 \implies H$ is contained in a parabolic subgroup P of G with $P/H = \mathbb{C}^*$.
- (2) $d_X = 2 \implies H$ is contained in a parabolic subgroup P of G with P/H being:
 - (a) \mathbb{C}
 - (b) the affine quadric Q_2
 - (c) the complement of a quadric curve in \mathbb{P}_2
 - (c) $(\mathbb{C}^*)^2$

2.5. Cousin group bundles over flag manifolds. In this section we prove a result concerning the structure of Kähler homogeneous manifolds whose normalizer fibrations are Cousin group bundles over flag manifolds, where there is no assumption about the invariant d . In order to do this we will show that one can reduce to the case where a complex reductive group is acting transitively and employ some now classical details about the structure of parabolic subgroups, e.g., see [8] or [16]. A crucial point in the setting of interest is the fact that all S -orbits are closed and have algebraic isotropy [20, Theorem 5.1].

Proposition 2.10. *Suppose G/H is a Kähler homogeneous manifold whose normalizer fibration $G/H \rightarrow G/N$ has fiber $C := N/H$ a Cousin group and base $Q := G/N$ a flag manifold. Then there exists a closed complex subgroup J of N containing H such that the fibration $G/H \rightarrow G/J$ realizes X as a $(\mathbb{C}^*)^k$ -bundle over a product $G/J = Q \times Y$, where Y is a Cousin group with $d_Y = d_X - k$.*

Proof. Our first task is to show that there is a reductive complex Lie group acting holomorphically and transitively on X . Write $C = \mathbb{C}^k/\Gamma$ and note that there exists a subgroup $\hat{\Gamma} < \Gamma$ such that $\hat{C} := \mathbb{C}^q/\hat{\Gamma}$ is isomorphic to $(\mathbb{C}^*)^q$ and is a covering group of C , e.g., see [1, §1.1]. In particular, the reductive complex Lie group $\hat{G} := S \times \hat{C}$ acts transitively on X . We drop the hats from now on and assume, by going to a finite covering, if necessary, that $G = S \times Z$ is a reductive complex Lie group, where $Z \cong (\mathbb{C}^*)^q$ is the center of G and S is a maximal semisimple subgroup.

Let $x_0 \in X$ be the neutral point, $z_0 \in G/N$ be its projection in the base, and F_{z_0} be the fiber over the point z_0 . Now the S -orbit $S \cdot H/H = S/S \cap H$ is closed in X and $S \cap H$ is an algebraic subgroup of S [20, Theorem 5.1]. So there is an induced fibration

$$(2.1) \quad \begin{array}{ccc} S/S \cap H & \hookrightarrow & G/H \\ A \downarrow & & \downarrow F_{z_0} \cong C \\ S/P & = & G/N \end{array}$$

Since the center normalizes any subgroup, we have $N = P \times Z$ with P a parabolic subgroup of S . Now any parabolic group P can be written as a semidirect product of its unipotent radical U_P and a subgroup L_P . Further, L_P is the centralizer in S of a subgroup C_P of L_P . In particular, it follows that C_P is the center of L_P , e.g., see [8]. In passing, we also note that the commutator subgroup of L_P is semisimple and thus L_P is a complex reductive Lie group.

The bundle in (2.1) is defined by a representation $\rho : N \rightarrow \text{Aut}^0(C) \cong C$ and the group $A = \rho(P)$ lies in the connected component of the identity of the automorphism group of C because P is connected. Since C is Abelian, $\rho|_P$ factors through the canonical projection from P to P/P' . Because every parabolic subgroup contains a maximal torus, every root space in the parabolic is in its commutator subgroup. Also C_P is not in the commutator subgroup, while the commutator subgroup of L_P is. As a consequence, $P/P' \cong (\mathbb{C}^*)^p$, with $p = \dim C_P$, e.g., see also [8, Proposition 8, §3.1]. Now the S -orbits in G/H are closed and the C_P -orbit in the typical fiber N/H through the neutral point is the intersection of N/H with the corresponding S -orbit. So this C_P -orbit is closed and complex and is an algebraic

quotient of the group C_P biholomorphic to $(\mathbb{C}^*)^k$ for some $k \leq p$. In this reductive setting S is a normal subgroup of G and so all S -orbits are biholomorphic and fiber over G/N with fiber biholomorphic to $(\mathbb{C}^*)^k$. If $k = 0$, then $X = C \times G/N$, by the argument given in the last paragraph of the proof.

Assume $k > 0$. Let $N \xrightarrow{\pi_1} N/H^0 \xrightarrow{\pi_2} N/H \cong C$, where π_1 is the canonical homomorphism with H^0 normal in N and π_2 is a covering homomorphism, since H/H^0 is a (normal) discrete subgroup of the Abelian group N/H^0 . Set

$$J := \pi_1^{-1} \circ \pi_2^{-1}(\rho(C_P)).$$

It follows that J is a closed complex Lie subgroup of G contained in N , since both π_1 and π_2 are holomorphic Lie group homomorphisms, that J contains H by its definition, and that the J -orbit through the neutral point is the C_P -orbit. This yields an intermediate fibration

$$G/H \xrightarrow{(\mathbb{C}^*)^k} G/J \longrightarrow G/N$$

with the first fiber $(\mathbb{C}^*)^k \cong A$.

We claim that the bundle $G/J \rightarrow G/N$ is holomorphically trivial. In order to see this note that the P -action on the neutral fiber of the bundle $G/J \rightarrow G/N$ is trivial. Otherwise, the dimension of the P -orbit in F_{z_0} would be bigger than k , as assumed above, and this would be a contradiction. As a consequence, all S -orbits in G/J are sections of the bundle $G/J \rightarrow G/N$. Since G/N is simply connected, this bundle is holomorphically trivial, i.e., $G/J = (C/A) \times S/P$. Because C is a Cousin group, $Y := C/A$ is also a Cousin group. The statement about the topological invariant follows because $d_{C/A} = d_{G/J}$, since S/P is compact, and $d_{G/J} = d_X - k$. \square

Remark 2.11. The case $d_X = 1$ is treated in [17, Proposition 5], where it is assumed that X has more than one end. For X Kähler this is equivalent to $d_X = 1$.

3. THE DISCRETE CASE

Throughout this section we assume that $X = G/\Gamma$ is Kähler with $d_X = 2$, where G is a connected, simply connected, complex Lie group and Γ is a discrete subgroup of G . We first show that G is solvable. Then we prove that a finite covering of such an X is biholomorphic to a product $C \times A$, where C is a Cousin group and A is a holomorphically separable complex Abelian Lie group.

3.1. The reduction to solvable groups. We first handle the case when the Kähler homogeneous manifold has no non-constant holomorphic functions.

Lemma 3.1. *Assume Γ is a discrete subgroup of a connected simply connected complex Lie group G that is not contained in a proper parabolic subgroup of G , with $X := G/\Gamma$ Kähler, $\mathcal{O}(X) = \mathbb{C}$, and $d_X \leq 2$. Then G is solvable.*

Proof. Assume $G = S \ltimes R$ is a Levi decomposition. Since the R -orbits are closed, we have a fibration

$$G/\Gamma \longrightarrow G/R \cdot \Gamma = S/\Lambda,$$

where $\Lambda := S \cap R \cdot \Gamma$ is Zariski dense and discrete in S , e.g., see [18]. Now if $\mathcal{O}(R \cdot \Gamma/\Gamma) = \mathbb{C}$, then the result was proved in [18]. Otherwise, let

$$R \cdot \Gamma/\Gamma \longrightarrow R \cdot \Gamma/J =: Y$$

be the holomorphic reduction. Then Y is holomorphically separable and since R acts transitively on Y , it follows that Y is Stein [24]. But $2 = d_X \geq d_Y \geq \dim_{\mathbb{C}} Y$. Further J^0 is a normal subgroup of G . As a consequence, the quotient group $\widehat{R} := R/J^0$ has complex dimension one or two. If $\dim \widehat{R} = 1$, then $\widehat{G} := S \times \widehat{R}$ is a product and this implies $S = \{e\}$ by [31]. If $\dim \widehat{R} = 2$, then \widehat{G} is either a product, see [31] again, or is a non-trivial semidirect product. In the latter case the result now follows by Theorem 2.8. \square

In the next Proposition we reduce to the setting where the Levi factor is $SL(2, \mathbb{C})$. We first prove a technical Lemma in that setting.

Lemma 3.2. *Suppose G/Γ is Kähler and $d_{G/\Gamma} \leq 2$, where Γ is a discrete subgroup of a connected, complex Lie group of the form $G = SL(2, \mathbb{C}) \ltimes R$ with R the radical of G . Then Γ is not contained in a proper parabolic subgroup of G .*

Proof. Assume the contrary, i.e., that Γ is contained in a proper parabolic subgroup and let P be a maximal such subgroup of G . Note that P is isomorphic to $B \ltimes R$, where B is a Borel subgroup of $SL(2, \mathbb{C})$. Let $P/\Gamma \rightarrow P/J$ be the holomorphic reduction. Then P/Γ is a Cousin group [32] and P/J is Stein [24]. Note that $J \neq P$, since otherwise P would be Abelian, a contradiction. The Fibration Lemma and Proposition 2.4 imply $\dim_{\mathbb{C}} P/J = 1$ or 2 . So P/J is biholomorphic to $\mathbb{C}, \mathbb{C}^*, \mathbb{C}^* \times \mathbb{C}^*$, or the complex Klein bottle.

In the first two cases P/J is equivariantly embeddable in \mathbb{P}_1 and by [26] it follows that G/J is Kähler. In the latter two cases the fiber J/Γ is compact by the Fibration Lemma and we can push down the Kähler metric on X to obtain a Kähler metric on G/J , e.g., see [12]. In particular, the S -orbit $S/(S \cap J)$ in G/J is Kähler and so its isotropy $S \cap J$ is algebraic [11]. Now consider the diagram

$$\begin{array}{ccccc} G/\Gamma & \xrightarrow{F} & G/J & \xrightarrow{Y} & G/P \\ & & & & \parallel \\ & \cup & \cup & & \mathbb{P}_1 \\ & & & & \parallel \\ S/S \cap \Gamma & \xrightarrow{F_S} & S/S \cap J & \xrightarrow{Z} & S/B \end{array}$$

Note that because $Y := P/J$ is noncompact and $d_{G/\Gamma} = 2$, it follows from the Fibration Lemma that either $d_F = 1$ or F is compact. Since F is an Abelian Lie group, it is clear that $d_{F_S} \leq d_F$.

The following enumerate, up to isomorphism, the algebraic subgroups of B and in each case we derive a contradiction.

- (1) $\dim_{\mathbb{C}} S \cap J = 2$. Then $S \cap J = B$. This yields the contradiction $d_{S/S \cap \Gamma} \leq d_{F_S} + d_{S/B} = 1 + 0 = 1 < 3 = d_{S/S \cap \Gamma}$, since $S \cap \Gamma$ is finite.
- (2) $\dim_{\mathbb{C}} S \cap J = 1$.
 - (a) If $S \cap J = \mathbb{C}^*$, then $S/S \cap J$ is an affine quadric or the complement of a quadric curve in \mathbb{P}_2 and thus $Z = \mathbb{C}$. So $P/J \neq \mathbb{C}^*$ and is either \mathbb{C} or $(\mathbb{C}^*)^2$, i.e., $d_{P/J} = 2$. Then the Fibration Lemma implies F is compact and, since F_S is closed in F , it must also be compact. But this forces $S \cap \Gamma$ to be an infinite subgroup of $S \cap J$ which is a contradiction.
 - (b) If $S \cap J = \mathbb{C}$, then $S/S \cap J$ is a finite quotient of $\mathbb{C}^2 \setminus \{(0, 0)\}$ and so $Z = \mathbb{C}^*$. Now $P/J = \mathbb{C}, (\mathbb{C}^*)^2$ or \mathbb{C}^* . In the first two instances F would

be compact and we get the same contradiction as in (a). In the last case $d_F = 1$ by the Fibration Lemma and F_S is either compact or \mathbb{C}^* . Again $S \cap \Gamma$ is infinite with the same contradiction as in (a).

- (3) $\dim_{\mathbb{C}} S \cap J = 0$. Here $S \cap J$ is finite, since it is an algebraic subgroup of B . Then $\dim S/S \cap J = 3$ and we see that $\dim G/J = 3$, since we know $\dim G/P = 1$ and $\dim P/J \leq 2$. Then $P/J = (\mathbb{C}^*)^2$ and, since $S/S \cap J$ is both open and closed in G/J , it follows that $S/S \cap J = G/J$ and $d_{S/S \cap J} = 2$. But F is compact and thus so is F_S and we get the contradiction that $d_{S/S \cap \Gamma} = 2 < 3 = d_{S/S \cap \Gamma}$.

As a consequence, Γ is not contained in a proper parabolic subgroup of G . \square

Proposition 3.3. *Suppose G/Γ is Kähler with $d_{G/\Gamma} \leq 2$. Then G is solvable.*

Proof. First note that G cannot be semisimple. If that were so, then Γ would be algebraic, hence finite and thus G/Γ would be Stein. But then $2 = d_{G/\Gamma} \geq \dim_{\mathbb{C}} G/\Gamma = \dim_{\mathbb{C}} G$ which is a contradiction, since necessarily $\dim_{\mathbb{C}} G \geq 3$ for any complex semisimple Lie group G .

So assume $G = S \ltimes R$ is mixed. The proof is by induction on the dimension of G . Now if a proper parabolic subgroup of G contains Γ , then a maximal one does too, is solvable by induction and so has the special form $B \ltimes R$, where B is isomorphic to a Borel subgroup of $S = SL(2, \mathbb{C})$. But this is impossible by Lemma 3.2.

Lemma 3.1 handles the case $\mathcal{O}(G/\Gamma) = \mathbb{C}$. So we assume $\mathcal{O}(G/\Gamma) \neq \mathbb{C}$ with holomorphic reduction $G/\Gamma \rightarrow G/J$. The Main Result in [7] gives the following list of the possibilities for the base G/J :

- (1) \mathbb{C}
- (2) affine quadric Q_2
- (3) $\mathbb{P}_2 \setminus Q$, where Q is quadric curve
- (4) an affine cone minus its vertex
- (5) \mathbb{C}^* -bundle over an affine cone minus its vertex

In case (1) the bundle is holomorphically trivial, with compact fiber a torus and the group that is acting effectively is solvable. In cases (2) and (3) we have fibrations $G/\Gamma \rightarrow G/J \rightarrow G/P = \mathbb{P}_1$ and so Γ is contained in a proper parabolic subgroup of G , contradicting what was shown in the previous paragraph.

In order to handle cases (4) and (5) we recall that an affine cone minus its vertex fibers equivariantly as a \mathbb{C}^* -bundle over a flag manifold. Thus we get fibrations

$$G/\Gamma \longrightarrow G/J \longrightarrow G/P.$$

Note that it cannot be the case that $G \neq P$, since then Γ would be contained in a proper parabolic subgroup, a possibility that has been ruled out. So $G = P$ and G/J (or a 2–1 covering) is biholomorphic to \mathbb{C}^* or $(\mathbb{C}^*)^2$. In the second case the fiber J/Γ is compact and thus a torus and thus G is solvable. If $G/J = \mathbb{C}^*$, then J/Γ is Kähler with $\dim J < \dim G$ and $d_{J/\Gamma} = 1$ by the Fibration Lemma. By induction J is solvable and so G is solvable too, because $G/J = \mathbb{C}^*$. \square

3.2. A product decomposition. In order to prove the classification we will have need of the next splitting result.

Proposition 3.4. *Suppose G is a connected, simply connected solvable complex Lie group that contains a discrete subgroup Γ such that G/Γ is Kähler and has holomorphic reduction $G/\Gamma \rightarrow G/J$ with base $(\mathbb{C}^*)^2$ and fiber a torus. Then a finite covering of G/Γ is biholomorphic to a product.*

Proof. First assume that J^0 is normal in G and let $\alpha : G \rightarrow G/J^0$ be the quotient homomorphism with differential $d\alpha : \mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{j}$. Then G/J^0 is a two dimensional complex Lie group that is either Abelian or solvable. In the Abelian case $G_0 := \alpha^{-1}(S^1 \times S^1)$ is a subgroup of G that has compact orbits in X , since these orbits fiber as torus bundles over $S^1 \times S^1$ in the base. The result now for (Γ, G_0, G) follows from [?, Theorem 6.14], since this triple is a CRS.

Next assume that G/J^0 is isomorphic to the two dimensional Borel group B with Lie algebra \mathfrak{b} . Then $\mathfrak{n} := d\alpha^{-1}(\mathfrak{n}_{\mathfrak{b}})$ is the nilradical of G . Let N denote the corresponding connected Lie subgroup of G . Now choose $\gamma_N \in \Gamma_N := N \cap \Gamma$ such that $\alpha(\gamma_N) \neq e$. There exists $x \in \mathfrak{n}$ such that $\exp(x) = \gamma_N$. Let U be the connected Lie group corresponding to $\langle \gamma_N \rangle_{\mathbb{C}}$. Since Γ centralizes J^0 (see [19, Theorem 1]), it follows that $\mathfrak{n} = \mathfrak{u} \oplus \mathfrak{j}$ and $N = U \times J^0$ is Abelian. Set $\Gamma_U := \Gamma \cap U$ and $\Gamma_J := \Gamma \cap J^0$. Then $N/\Gamma_N = U/\Gamma_U \times J^0/\Gamma_J$.

Since $\Gamma/\Gamma_N = \mathbb{Z}$, we may choose $\gamma \in \Gamma$ such that γ projects to a generator of Γ/Γ_N . Also set $A := \exp(\langle w \rangle_{\mathbb{C}})$ for fixed $w \in \mathfrak{g} \setminus \mathfrak{n}$. Since A is complementary to N , we have $G = A \ltimes N$. Now there exist $\gamma_A \in A$ and $\gamma_N \in N$ such that $\gamma = \gamma_A \cdot \gamma_N$. Both γ and γ_N centralize J^0 and thus γ_A does too. Also $\gamma_A = \exp(h)$ for some $h = sw$ with $s \in \mathbb{C}$. Therefore,

$$(3.1) \quad [h, \mathfrak{j}] = 0.$$

Since $\mathfrak{a} + \mathfrak{u}$ is isomorphic to $\mathfrak{b} = \mathfrak{g}/\mathfrak{j}$ as a vector space, there exists $e \in \mathfrak{u}$ such that

$$[d\alpha(h), d\alpha(e)] = 2d\alpha(e).$$

Let $\{e_1, \dots, e_{n-2}\}$ be a basis for \mathfrak{j} . There exist structure constants a_i so that

$$[h, e] = 2e + \sum_{i=1}^{n-2} a_i e_i$$

and the remaining structure constants are all 0 by (3.1). Note that, conversely, any choice of the structure constants a_i determines a solvable Lie algebra \mathfrak{g} and the corresponding connected simply-connected complex Lie group $G = A \ltimes N$.

We now compute the action of $\gamma_A \in A$ on N by conjugation. In order to do this note that the adjoining representation restricted to \mathfrak{n} , i.e., the map $\text{ad}_h : \mathfrak{n} \rightarrow \mathfrak{n}$, is expressed by the matrix

$$M := [\text{ad}_h] = \begin{pmatrix} 2 & 0 & \dots & 0 \\ a_1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-2} & 0 & \dots & 0 \end{pmatrix}$$

So the action of A on N is through the one parameter group of linear transformations $t \mapsto e^{tM}$ for $t \in \mathbb{C}$. For $k \geq 1$

$$(tM)^k = \frac{1}{2}(2t)^k M,$$

and it follows that

$$e^{tM} = \frac{1}{2}(e^{2t} - 1)M + \text{Id}.$$

Now the projection of the element γ_A acts trivially on the base $Y = G/J$, so $t = \pi i k$ where $k \in \mathbb{Z}$. Hence γ_A acts trivially on U . Also γ_N acts trivially on N , since N is Abelian. Thus γ acts trivially on N and as a consequence, although G is a non-Abelian solvable group the manifold $X = G/\Gamma$ is just the quotient of \mathbb{C}^n by a discrete additive subgroup of rank $2n - 2$. Its holomorphic reduction is the original torus bundle which, since we are now dealing with the Abelian case, is trivial.

Now assume J^0 is not normal in G , set $N := N_G(J^0)$, and let $G/J \xrightarrow{N/J} G/N$ be the normalizer fibration. Since the base G/N of the normalizer fibration is an orbit in some projective space, G/N is holomorphically separable and thus Stein [24]. Since we also have $2 \geq d_{G/N} \geq \dim_{\mathbb{C}} G/N$ we see that $G/N \cong \mathbb{C}$, \mathbb{C}^* or $\mathbb{C}^* \times \mathbb{C}^*$. We claim that we must have $G/N = \mathbb{C}^*$, i.e., we have to eliminate the other two possibilities. Assume $G/N \cong \mathbb{C}$. Since $d_X \leq 2$ the Fibration Lemma implies $d_{N/J} = 0$, i.e., N/J is biholomorphic to a torus T . Thus $G/J = T \times \mathbb{C}$. However, we assume that $G/J = \mathbb{C}^* \times \mathbb{C}^*$ and this gives us a contradiction. Now assume $G/N \cong \mathbb{C}^* \times \mathbb{C}^*$. By Chevalley's theorem [14] the commutator group G' acts algebraically. Hence the G' -orbits are closed and one gets the commutator fibration $G/N \xrightarrow{\mathbb{C}} G/G' \cdot N$. Since G is solvable, it follows that G' is unipotent and the G' -orbits are cells, i.e., $G' \cdot N/N \cong \mathbb{C}$. By the Fibration Lemma the base of the commutator fibration is a torus. But it is proved in [23] that the base of a commutator fibration is always Stein which is a contradiction. This proves the claim that $G/N \cong \mathbb{C}^*$ and by the Fibration Lemma $d_{N/J} = 1$ and hence $N/J = \mathbb{C}^*$.

Since G is simply connected, G admits a Hochschild-Mostow hull ([22]), i.e., there exists a solvable linear algebraic group

$$G_a = (\mathbb{C}^*)^k \ltimes G$$

that contains G as a Zariski dense, topologically closed, normal complex subgroup. By passing to a subgroup of finite index we may, without loss of generality, assume the Zariski closure $G_a(\Gamma)$ of Γ in G_a is connected. Then $G_a(\Gamma) \supseteq J^0$ and $G_a(\Gamma)$ is nilpotent [19]. Let $\pi : \widehat{G}_a(\Gamma) \rightarrow G_a(\Gamma)$ be the universal covering and set $\widehat{\Gamma} := \pi^{-1}(\Gamma)$. Since $\widehat{G}_a(\Gamma)$ is a simply connected, nilpotent, complex Lie group, the exponential map from the Lie algebra $\mathfrak{g}_a(\Gamma)$ to $\widehat{G}_a(\Gamma)$ is bijective. For any subset of $\widehat{G}_a(\Gamma)$ and, in particular for the subgroup $\widehat{\Gamma}$, the smallest closed, connected, complex (resp. real) subgroup \widehat{G}_1 (resp. \widehat{G}_0) of $\widehat{G}_a(\Gamma)$ containing $\widehat{\Gamma}$ is well-defined. By construction $\widehat{G}_0/\widehat{\Gamma}$ is compact, e.g., see [33, Theorem 2.1]. Set $G_1 := \pi(\widehat{G}_1)$ and $G_0 := \pi(\widehat{G}_0)$ and consider the CRS manifold (G_1, G_0, Γ) . Note that the homogeneous CR-manifold G_0/Γ fibers as a T -bundle over $S^1 \times S^1$. In order to understand

the complex structure on the base $S^1 \times S^1$ of this fibration consider the diagram

$$\begin{array}{ccccc}
\widehat{G}_0/\widehat{\Gamma} & \subset & \widehat{G}_1/\widehat{\Gamma} & \subseteq & \widehat{G}_a(\Gamma)/\widehat{\Gamma} \\
\parallel & & \parallel & & \parallel \\
G_0/\Gamma & \subset & G_1/\Gamma & \subseteq & G_a(\Gamma)/\Gamma \\
T \downarrow & & T \downarrow & & \downarrow T \\
S^1 \times S^1 = G_0/G_0 \cap (J^0 \cdot \Gamma) & \subset & G_1/J^0 \cdot \Gamma & \subseteq & G_a/J^0 \cdot \Gamma
\end{array}$$

As observed in [19, Theorem 1], the manifold $G_a/J^0 \cdot \Gamma$ is a holomorphically separable solvmanifold and thus is Stein [24]. So $G_1/J^0 \cdot \Gamma$ is also Stein and thus $G_0/G_0 \cap (J^0 \cdot \Gamma)$ is totally real in $G_1/J^0 \cdot \Gamma$. The corresponding complex orbit $G_1/J^0 \cdot \Gamma$ is then biholomorphic to $\mathbb{C}^* \times \mathbb{C}^*$. It now follows by [19, Theorem 6.14] that a finite covering of G_1/Γ splits as a product of a torus with $\mathbb{C}^* \times \mathbb{C}^*$ and, in particular, that a subgroup of finite index in Γ is Abelian.

Now set $A := \{ \exp t\xi \mid t \in \mathbb{C} \}$, where $\xi \in \mathfrak{g} \setminus \mathfrak{n}$ and \mathfrak{n} is the Lie algebra of N^0 . Then $G = A \ltimes N^0$ and any $\gamma \in \Gamma$ can be written as $\gamma = \gamma_A \cdot \gamma_N$ with $\gamma_A \in A$ and $\gamma_N \in N$. The fiber $G/\Gamma \rightarrow G/N$ is the N^0 -orbit of the neutral point and Γ acts on it by conjugation. Since N/Γ is Kähler and has two ends, it follows by [17, Proposition 1] that a finite covering of N/Γ is biholomorphic to a product of the torus and \mathbb{C}^* . (By abuse of language we still call the subgroup having finite index Γ .) Now the bundle $G/\Gamma \rightarrow G/N$ is associated to the bundle

$$\mathbb{C} = G/N^0 \longrightarrow G/N = \mathbb{C}^*$$

and thus $G/\Gamma = \mathbb{C} \times_{\rho} (T \times \mathbb{C}^*)$, where $\rho : N/N^0 \rightarrow \text{Aut}(T \times \mathbb{C}^*)$ is the adjoint representation. Since Γ is Abelian, this implies γ_A acts trivially on $\Gamma_N := \Gamma \cap N^0$. Now suppose J has complex dimension k . Then γ_A is acting as a linear map on $N^0 = \mathbb{C} \ltimes J^0 = \mathbb{C}^{k+1}$ and commutes with the additive subgroup Γ_N that has rank $2k+1$ and spans N^0 as a linear space. This implies γ_A acts trivially on N^0 and, as a consequence, the triviality of a finite covering of the bundle, as required. \square

3.3. The classification for discrete isotropy. In the following we classify Kähler G/Γ when Γ is discrete and $d_X \leq 2$. Note that $d_X = 0$ means X is compact and this is the now classical result of Borel–Remmert [13] and the case $d_X = 1$ corresponds to X having more than one end and was handled in [17].

Theorem 3.5 ([4]). *Let G be a connected simply connected complex Lie group and Γ a discrete subgroup of G such that $X := G/\Gamma$ is Kähler and $d_X \leq 2$. Then G is solvable and a finite covering of X is biholomorphic to a product $C \times A$, where C is a Cousin group and A is $\{e\}, \mathbb{C}^*, \mathbb{C}$, or $(\mathbb{C}^*)^2$. Moreover, $d_X = d_C + d_A$.*

Proof. By Proposition 3.3 G is solvable. If $\mathcal{O}(X) \cong \mathbb{C}$, then X is a Cousin group [32]. Otherwise, $\mathcal{O}(X) \neq \mathbb{C}$ and let

$$G/\Gamma \xrightarrow{J/\Gamma} G/J$$

be the holomorphic reduction. Its base G/J is Stein [24], its fiber J/Γ is biholomorphic to a Cousin group [32], and a finite covering of the bundle is principal [19].

Since G/J is Stein, by Proposition 2.4 one has

$$\dim_{\mathbb{C}} G/J \leq d_{G/J} \leq d_X \leq 2.$$

If $d_X = 1$, then $d_{G/J} = 1$ and G/J is biholomorphic to \mathbb{C}^* . A finite covering of this bundle is principal, with the connected Cousin group as structure group, and so is holomorphically trivial [21]. If $d_X = 2$, the Fibration Lemma implies $G/J \cong \mathbb{C}$, \mathbb{C}^* , $\mathbb{C}^* \times \mathbb{C}^*$ or a complex Klein bottle [7]. The case of \mathbb{C}^* is handled as above and a torus bundle over \mathbb{C} is trivial by Grauert's Oka Principle [21]. Finally, since a Klein bottle is a 2-1 cover of $\mathbb{C}^* \times \mathbb{C}^*$, it suffices to consider the case $\mathbb{C}^* \times \mathbb{C}^*$. That case is handled by Proposition 3.4. \square

4. PROOF OF THE MAIN RESULT WHEN $\mathcal{O}(X) = \mathbb{C}$

Proof. Let $\pi : G/H \rightarrow G/N$ be the normalizer fibration. We claim that G' acts transitively on $Y := G/N$. In order to see this consider the commutator fibration $G/N \rightarrow G/N \cdot G'$ which exists by Chevalley's theorem, see [14], and recall that its base $G/N \cdot G'$ is an Abelian affine algebraic group that is Stein [23] and so must be a point. Otherwise, one could pullback non-constant holomorphic functions to X contradicting the assumption $\mathcal{O}(X) = \mathbb{C}$. Since G' acts on G/N as an algebraic group of transformations and $d_{G/N} \leq d_X = 2$, there is a parabolic subgroup P of G' containing $N \cap G'$ (see [6] or Theorem 2.9) and we now consider the fibration $Y = G'/N \cap G' \rightarrow G'/P$.

First we assume G/N is compact and thus a flag manifold, i.e., $N \cap G' = P$ and suppose $\mathcal{O}(N/H) = \mathbb{C}$. Then N/H is a Cousin group by Theorem 3.5. The structure in this case is given in Proposition 2.10: X fibers as a $(\mathbb{C}^*)^k$ -bundle over a product $C \times Q$ with $d_C + k = 2$ with C a Cousin group and Q a flag manifold.

Next suppose G/N compact and $\mathcal{O}(N/H) \neq \mathbb{C}$ with holomorphic reduction $N/H \rightarrow N/I$. The possibilities for N/H have been presented in Theorem 3.5; N/H is a solvmanifold, N/I is Stein [24] and I/H is a Cousin group [32]. So we get the following cases and we claim that none of these can actually occur:

- (i) $N/I = \mathbb{C}^*$ and $I/H =: C$ is a Cousin group of hypersurface type.
- (ii) $N/I = (\mathbb{C}^*)^2$ and $I/H = T$ is a torus.
- (iii) $N/I = \mathbb{C}$ and $I/H = T$ is a torus.

In (i) and (ii) we have $\mathcal{O}(G/I) \neq \mathbb{C}$ which contradicts $\mathcal{O}(X) = \mathbb{C}$. The same contradiction occurs in (iii). Let $I := \text{Stab}_G(T)$. Then N normalizes I , since the partition of $T \times \mathbb{C}$ by the cosets of T is N -invariant. Thus $N/I \cong \mathbb{C}$ as groups. Now S is not transitive on G/I since this would give an affine bundle $S/S \cap I \rightarrow S/P$ with $S \cap I$ not normal in P [10, Proposition 1 in §5.2]. Therefore the S -orbits in G/I are sections, the bundle $G/I \rightarrow G/N$ is holomorphically trivial and $\mathcal{O}(G/I) \neq \mathbb{C}$. However, this once again contradicts $\mathcal{O}(X) = \mathbb{C}$.

Now suppose $d_{G/N} = 1$. As noted above, G' acts algebraically and transitively on G/N and G/N is an affine cone minus its vertex or simply \mathbb{C}^* . Thus $\mathcal{O}(G/N) \neq \mathbb{C}$ contradicting $\mathcal{O}(X) = \mathbb{C}$.

Suppose $d_{G/N} = 2$ and a finite covering of $P/N \cap G'$ is biholomorphic to $(\mathbb{C}^*)^2$. An intermediate fibration has base an affine cone minus its vertex and there exist non-constant holomorphic functions with the same contradiction as before.

Finally, suppose $d_{G/N} = 2$ and $P/N \cap G' = \mathbb{C}$. Then there are two possibilities and we first suppose S acts transitively on Y . Let x_0 be any point in X and $z_0 \in S/I$ be its projection and consider the S -orbit $S.x_0 = S/S \cap H$ through the point x_0 . There is an induced fibration

$$\begin{array}{ccc} S/S \cap H & \hookrightarrow & G/H \\ A \downarrow & & \downarrow F_{z_0} \\ S/I & = & G/N \end{array}$$

with I an algebraic subgroup of S and $F_{z_0} = T$ a compact complex torus by the Fibration Lemma. Since $S/S \cap H$ is Kähler, $S \cap H$ is an algebraic subgroup of S . Then the holomorphic bundle $S/S \cap H \rightarrow S/I$ has the algebraic variety $I/S \cap H$ as fiber A and this is a closed subgroup of the torus T . But this is only possible if $I/S \cap H$ is finite. Since we have the fibration $S/I \rightarrow S/P$ with $P/I = \mathbb{C}$ and S/P a flag manifold, we see that S/I is simply connected. As a consequence, every S -orbit in X is a holomorphic section and we conclude that X is the product of T and S/I .

Otherwise, S does not act transitively on G/N . The radical $R_{G'}$ of G' is a unipotent group acting algebraically on G/N yielding a fibration

$$Y = G/N \xrightarrow{F} G/N \cdot R_{G'}$$

where $F = \mathbb{C}^p$ with $p > 0$. The Fibration Lemma and the assumption $d_X = 2$ imply N/H is compact, thus a torus, $F = \mathbb{C}$ and $Z := G/N \cdot R_{G'}$ is compact and thus a flag manifold. Now $Y \neq \mathbb{C} \times Z$ because one would then have $\mathcal{O}(Y) \neq \mathbb{C}$ contrary to $\mathcal{O}(X) = \mathbb{C}$. So Y is a non-trivial line bundle over Z and there are two S -orbits in Y , a compact one Y_1 which is the zero section of the line bundle and is biholomorphic to Z and an open one Y_2 . The latter holds, since the existence of another closed orbit would imply the triviality of the \mathbb{C}^* -bundle $Y \setminus Y_1$ over Z . We write X as a disjoint union $X_1 \cup X_2$ with $X_i := \pi^{-1}(Y_i)$ for $i = 1, 2$. Then X_1 is a Kähler torus bundle over Z and is trivial by [13]. And a finite covering of X_2 is also trivial since X_2 is Kähler and satisfies $d_{X_2} = 1$ [17, Main Theorem, Case (b)]. Note that the S -orbits in X_1 (resp. X_2) are holomorphic sections of the torus bundle lying over the corresponding S -orbit Y_1 (resp. Y_2).

Let $x_2 \in X_2$ and $M_2 := S.x_2$. Since X is Kähler, the boundary of M_2 consists of S -orbits of strictly smaller dimension [20, Theorem 3.6], and for dimension reasons, these necessarily lie in X_1 . Let $M_1 \subset \overline{M_2}$ be such an S -orbit in X_1 and let $p \in M_1$. As observed in the previous paragraph, M_1 is biholomorphic to Y_1 which is a flag manifold. Therefore, $M_1 = K.p = K/L$, where K is a maximal compact subgroup of S and L is the corresponding isotropy subgroup at the point p and is compact. The L -action at the L -fixed point p can be linearized, i.e. this L -action leaves invariant the complex vector subspaces $T_p(K/L)$ and $T_p(\pi^{-1}(\pi(p)))$ as well as a complementary complex vector subspace W of $T_p(X)$. Now $W \oplus T_p(K/L)$ is the tangent space to $\overline{M_2}$ and is a complex vector space. This implies that M_2 is the unique S -orbit that contains M_1 in its closure and also that $\overline{M_2} = M_2 \cup M_1$ is a complex submanifold of X that is a holomorphic section of the bundle $\pi : X \rightarrow Y$. This bundle is thus trivial and X is biholomorphic to the product $T \times Y$.

This completes the classification if $\mathcal{O}(X) = \mathbb{C}$. □

5. PROOF OF THE MAIN RESULT WHEN $\mathcal{O}(X) \neq \mathbb{C}$

We first prove a generalization of Proposition 3.4 for arbitrary isotropy.

Proposition 5.1. *Let G be a connected, simply connected, solvable complex Lie group and H a closed complex subgroup of G with G/H Kähler and $G/H \rightarrow G/J$ its holomorphic reduction with fiber $T = J/H$ a compact complex torus and base $G/J = (\mathbb{C}^*)^2$. Then a finite covering of G/H is biholomorphic to $T \times (\mathbb{C}^*)^2$.*

Proof. If H^0 is normal in G , then this is Proposition 3.4. Otherwise, let $N := N_G(H^0)$ and consider $G/H \rightarrow G/N$. Because G/N is an orbit in some projective space, G/N is holomorphically separable and the map $G/H \rightarrow G/N$ factors through the holomorphic reduction, i.e., $J \subset N$. We first assume that $J = N$ and consider now $\hat{N} := N_G(J^0)$. Then the argument given in the third last paragraph of the proof of Proposition 3.4 shows that $G/\hat{N} = \mathbb{C}^*$ and $\hat{N}/J = \mathbb{C}^*$. But then (a finite covering of) \hat{N}/H is isomorphic to $\mathbb{C}^* \times T$, see [17, Proposition 1], i.e., that H^0 is normal in \hat{N} . This gives the contradiction that $\hat{N} = N$ while $\dim \hat{N} > \dim J = \dim N$.

So we are reduced to the setting where, after going to a finite covering if necessary, $G/N = \mathbb{C}^*$ and $N/H \cong \mathbb{C}^* \times T$ is an Abelian complex Lie group, since N/H is Kähler with two ends, see [17, Proposition 1]. We have the diagram

$$\begin{array}{ccc} X = G/H & \xrightarrow{T} & G/J = \mathbb{C}^* \times \mathbb{C}^* \\ & \searrow & \swarrow \\ & G/N_G(H^0) = \mathbb{C}^* & \end{array}$$

Since the top line is the holomorphic reduction and X is Kähler, a finite covering of this bundle is a T -principal bundle, see [19, Theorem 1]. Choose $\xi \in \mathfrak{g} \setminus \mathfrak{n}$ and set $A := \exp\langle \xi \rangle_{\mathbb{C}}$ and $B := N_G(H^0)/H \cong \mathbb{C}^* \times T$. Then the group $\hat{G} := A \ltimes B$ acts holomorphically and transitively on X , where A acts from the left and B acts from the right on the principal $\mathbb{C}^* \times T$ -bundle $G/H \rightarrow G/N$. For dimension reasons the isotropy of this action is discrete and the result now follows by Proposition 3.4. \square

Proof of the Main Result when $\mathcal{O}(X) \neq \mathbb{C}$. Let $G/H \rightarrow G/J$ be the holomorphic reduction. In [7] there is a list of the possibilities for G/J which was also given above in the proof of Proposition 3.3. We now employ that list to determine the structure of X .

Suppose $G/J = \mathbb{C}$. By the Fibration Lemma J/H is compact, Kähler, and so biholomorphic to the product of a torus T and a flag manifold Q . Thus $X = T \times Q \times \mathbb{C}$ by [21].

Suppose G/J is an affine quadric. By the Fibration Lemma we again have $J/H = T \times Q$. Then X is biholomorphic to a product, since G/J is Stein and is simply connected [21].

If G/J is the complement of the quadric curve in \mathbb{P}_2 , then a two-to-one covering of G/J is the affine quadric and the pullback of X to that space is again a product, as in the previous case.

Suppose G/J is a \mathbb{C}^* -bundle over an affine cone minus its vertex and so $d_{G/J} = 2$ and let $G = S \ltimes R$ be a Levi–Malcev decomposition of G . By the Fibration Lemma

J/H is compact and J/H inherits a Kähler structure from X . If we set $N := N_J(H^0)$, then the normalizer fibration $J/H \rightarrow J/N$ is a product with $N/H = T$ and $J/N = Q$, i.e., $J/H = T \times Q$, see [13]. Now S is acting transitively on G/N and by pulling the bundle back to the universal covering of G/J , without loss of generality, we may assume that G/N is simply connected. Consider the bundle $G/H \rightarrow G/N$. Its fibers are biholomorphic to T and the intersection of every S -orbits with this fiber N/H is biholomorphic to Q . Since the S -orbits map onto G/J , it follows that the S -orbits are sections of the bundle $G/H \rightarrow G/N$ and the bundle is holomorphically trivial. Hence a finite covering of X is biholomorphic to $T \times (S/S \cap N)$. Example 6.5 shows that the Q -bundle $S/S \cap N \rightarrow S/S \cap J$ is not necessarily trivial.

Suppose the base of the holomorphic reduction of G/H is $G/J = (\mathbb{C}^*)^2$, i.e., there is no flag manifold involved. We assume that G is simply connected and has a Levi–Malcev decomposition $G = S \ltimes R$. Let $p : G/H \rightarrow G/J$ be the bundle projection map. We now fiber J/H in terms of its S -orbits first. Then the partition of X by the fibers of the map p is S -invariant. And J/H is compact and Kähler and thus biholomorphic to a product $T \times Q$, where the flag manifold passing through each point is an S -orbit. As a consequence, the S -orbit in X are all closed and biholomorphic to Q and we have a homogeneous fibration $G/H \rightarrow G/S \cdot H$. Since all holomorphic functions are constant on the fibers of this fibration, the holomorphic reduction factors through this bundle projection and the base $G/S \cdot H = R/R \cap H$ then fibers as T -bundle over the base of the holomorphic reduction G/J and is Kähler [12]. As a consequence of Proposition 5.1, $X = Q \times T \times (\mathbb{C}^*)^2$.

Finally, suppose G/J is an affine cone minus its vertex (or possibly \mathbb{C}^*) and thus $d_{G/J} = 1$. Now by the Fibration Lemma the fiber J/H satisfies $d_{J/H} = 1$ and is Kähler. The classification given in [17, Main Theorem] now applies. First assume $\mathcal{O}(J/H) = \mathbb{C}$. Then by [17, Proposition 5] the normalizer fibration $J/H \rightarrow J/N$ realizes J/H as a Cousin bundle over a flag manifold. By Proposition 2.10 there is a closed complex subgroup I of J containing H with $I/H = \mathbb{C}^*$ and $J/I = T \times Q$, a product of a torus and a flag manifold. The S -orbits in G/I are sections and the torus bundle splits as a product $Y := T \times S/S \cap I$. Thus X fibers as a \mathbb{C}^* -bundle over $T \times Y$. Example 6.4 shows that Y itself need not be a product. Next assume $\mathcal{O}(J/H) \neq \mathbb{C}$. Then by [17, Proposition 3] there exists a closed complex subgroup I of J containing H such that a finite covering of J/H is biholomorphic to $I/H \times J/I$, where $I/H = T$ is a torus and $Z := J/I$ is an affine cone minus its vertex. Then the fibration $G/H \rightarrow G/I$ has T as fiber and S is transitive on its base and since T is compact, there is a Kähler structure on G/I , see [12]. Now by the Fibration Lemma we have $d_{G/I} = 2$ and thus there exists a parabolic subgroup P of G containing I such that P/I is isomorphic to $(\mathbb{C}^*)^2$, see [6] or Theorem 2.9. But then G/I is holomorphically separable (G/I can be realized as a \mathbb{C}^* -bundle over an affine cone minus its vertex) and so G/I is the base of the holomorphic reduction of G/H . But this contradicts the assumption that G/J is the base of the holomorphic reduction of G/H . Note that it is not possible that $J = I$, since $\mathcal{O}(J/H) \neq \mathbb{C}$ implies $\dim J/I > 0$. This case does not occur.

This completes the classification when $\mathcal{O}(X) \neq \mathbb{C}$. □

6. EXAMPLES

We now give non-trivial examples that can occur in the classification.

Example 6.1. The manifolds that occur in Proposition 2.10 need not be biholomorphic to a product of an S -orbit times an orbit of the center. For $k = d_X = 1$, let $\chi : B \rightarrow \mathbb{C}^*$ be a non-trivial character, where B is a Borel subgroup of $S := SL(2, \mathbb{C})$. Let C be a non-compact 2-dimensional Cousin group. Then C fibers as a \mathbb{C}^* -bundle over an elliptic curve T and let B act on C via the character χ . Set $X := S \times_B C$. Then X fibers as a principal C -bundle over S/B and is Kähler, but neither this bundle nor the corresponding \mathbb{C}^* -bundle is trivial.

Example 6.2. Let $S := SL(3, \mathbb{C})$ and

$$H := \left\{ \begin{pmatrix} * & 0 & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \right\} \subset B \subset P := \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \right\},$$

where B is the Borel subgroup of S consisting of upper triangular matrices. Then $S/H \rightarrow S/B$ is an affine \mathbb{C} -bundle over the flag manifold S/B . Now consider the fibration $S/H \rightarrow S/P$. Its fiber is $P/H = \mathbb{P}_2 \setminus \{\text{point}\}$ and all holomorphic functions on S/H are constant along the fibers by Hartogs' Principle and so must come from the base $S/P = \mathbb{P}_2$. But the latter is compact and so $\mathcal{O}(S/P) = \mathbb{C}$ and, as a consequence, we see that $\mathcal{O}(S/H) = \mathbb{C}$. Thus S/H is an example of a space that can be the base of the normalizer fibration in the second paragraph of the proof of the Main Theorem in §4.

Example 6.3. The space $Y = \mathbb{P}_n \setminus \{z_0\}$ is an example for the base of the normalizer fibration in the third paragraph of the proof of the Main Theorem in §4. For $n = 2$ we have $Y = P/H$ with the groups given in Example 6.2.

Example 6.4. Consider the subgroups of $S := SL(5, \mathbb{C})$ defined by

$$H := \left\{ \begin{pmatrix} 1 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \end{pmatrix} \right\} \subset P := \left\{ \begin{pmatrix} * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \end{pmatrix} \right\}$$

and

$$J := P' = \left\{ \begin{pmatrix} 1 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \end{pmatrix} \right\}$$

Then $J/H = \mathbb{P}_1$ and $P/J = \mathbb{C}^*$ and $S/P = Q$ is a flag manifold that can be fibered as a $\text{Gr}(2, 4)$ -bundle over \mathbb{P}_4 . We have the fibrations

$$S/H \xrightarrow{\mathbb{P}_1} S/J \xrightarrow{\mathbb{C}^*} S/P = Q.$$

Note that S/J is holomorphically separable due to the fact that it can be equivariantly embedded as an affine cone minus its vertex in some projective space and since J/H is compact, S/J is the base of the holomorphic reduction of S/H . Because the fibration of S/H is not trivial, this shows that the S -orbit that can occur in the proof in §5 need not split as a product.

Example 6.5. One can take a parabolic subgroup similar to the one in the previous example so that its center has dimension two and create an example which fibers as a non-trivial flag manifold over a $(\mathbb{C}^*)^2$ -bundle over a flag manifold. Since this is similar to the construction in Example 6.4, we leave the details to the reader.

REFERENCES

- [1] Y. Abe and K. Kopfermann, *Toroidal groups. Line bundles, cohomology and quasi-abelian varieties*. Lecture Notes in Mathematics, 1759. Springer-Verlag, Berlin, 2001.
- [2] H. Abels, *Proper transformation groups*. In: “Transformation groups. Proc. Conf., Univ. Newcastle upon Tyne, Newcastle upon Tyne, 1976”, pp. 237 – 248. London Math. Soc. Lecture Note Series, No. 26, Cambridge Univ. Press, Cambridge, 1977.
- [3] H. Abels, *Some topological aspects of proper group actions; noncompact dimension of groups*. J. London Math. Soc. (2) **25** (1982), no. 3, 525 – 538.
- [4] S. Ahmadi, *A classification of homogeneous Kähler manifolds with discrete isotropy and top non-vanishing homology in codimension two*. Ph.D. Dissertation, University of Regina, 2013.
- [5] D. N. Ahiezer, *Dense orbits with two endpoints*. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. **41** (1977), no. 2, 308 – 324; translation in Math. USSR-Izv. **11** (1977), no. 2, 293 – 307 (1978).
- [6] D. N. Akhiezer, *Complex n -dimensional homogeneous spaces homotopically equivalent to $(2n - 2)$ -dimensional compact manifolds*. Selecta Math. Sov. **3**, no.3 (1983/84), 286 – 290.
- [7] D. N. Akhiezer and B. Gilligan, *On complex homogeneous spaces with top homology in codimension two*, Canad. J. Math. **46** (1994), 897 – 919.
- [8] D. N. Akhiezer, “Lie Group Actions in Complex Analysis”, Aspects of Mathematics, Friedr. Vieweg & Sohn Verlagsgesellschaft mbH, Braunschweig/Wiesbaden, 1995.
- [9] L. Auslander and R. Tolimieri, *Splitting theorems and the structure of solvmanifolds*. Ann. of Math. (2) **92** (1970), 164 – 173.
- [10] H. Azad, A. Huckleberry, and W. Richthofer, *Homogeneous CR-manifolds*. J. Reine Angew. Math. **358** (1985), 125 – 154.
- [11] R. Berteloot and K. Oeljeklaus, *Invariant plurisubharmonic functions and hypersurfaces on semi-simple complex Lie groups*. Math. Ann. **281** (1988), no. 3, 513 – 530.
- [12] A. Blanchard, *Sur les variétés analytiques complexes*. Ann. Sci. Ecole Norm. Sup. (3) **73** (1956), 157 – 202.
- [13] A. Borel and R. Remmert, *Über kompakte homogene Kählersche Mannigfaltigkeiten*. (German) Math. Ann. **145** (1961/1962), 429 – 439.
- [14] C. Chevalley, “Théorie des Groupes de Lie II. Groupes Algébriques”, Hermann, Paris, 1951.
- [15] J. Dorfmeister and K. Nakajima, *The fundamental conjecture for homogeneous Kähler manifolds*. Acta Math. **161** (1988), no. 1-2, 23 – 70.
- [16] G. Fels, A. Huckleberry, and J. Wolf, “Cycle spaces of flag domains. A complex geometric viewpoint”. Progress in Mathematics, **245**. Birkhäuser Boston, Inc., Boston, MA, 2006.
- [17] B. Gilligan, K. Oeljeklaus, and W. Richthofer, *Homogeneous complex manifolds with more than one end*. Canad. J. Math. **41** (1989), no. 1, 163 – 177.
- [18] B. Gilligan, *Invariant analytic hypersurfaces in complex Lie groups*, Bull. Austral. Math. Soc. **70** (2004), no. 2, 343 – 349.
- [19] B. Gilligan and K. Oeljeklaus, *Two remarks on Kähler homogeneous manifolds*, Ann. Fac. Sci. Toulouse Math. (6) **17** (2008), 73 – 80.
- [20] B. Gilligan, C. Miebach, and K. Oeljeklaus, *Homogeneous Kähler and Hamiltonian manifolds*, Math. Ann. **349** (2011), 889 – 901.
- [21] H. Grauert, *Analytische Faserungen über holomorph-vollständigen Räumen*. Math. Ann. **135** (1958), 263 – 273.
- [22] G. Hochschild and G. D. Mostow, *On the algebra of representative functions of an analytic group. II*, Amer. J. Math., **86** (1964), 869 – 887.
- [23] A. T. Huckleberry and E. Oeljeklaus, *Homogeneous Spaces from a Complex Analytic Viewpoint*, In: Manifolds and Lie groups. (Papers in honor of Y. Matsushima), Progress in Math., **14**, Birkhäuser, Boston (1981), 159 – 186.
- [24] A. T. Huckleberry and E. Oeljeklaus, *On holomorphically separable complex solv-manifolds*. Ann. Inst. Fourier (Grenoble) **36** (1986), no. 3, 57 – 65.

- [25] J. E. Humphreys, “Linear algebraic groups”. Graduate Texts in Mathematics, No. 21. Springer-Verlag, New York-Heidelberg, 1975.
- [26] K. Kodaira, *On Kähler varieties of restricted type (an intrinsic characterization of algebraic varieties)*. Ann. of Math. (2) **60** (1954), 28 – 48.
- [27] J.-J. Loeb, *Action d’une forme réelle d’un groupe de Lie complexe sur les fonctions plurisousharmoniques*. Ann. Inst. Fourier (Grenoble) **35** (1985), 59 – 97.
- [28] Y. Matsushima, *Sur les espaces homogènes kählériens d’un groupe de Lie réductif*. (French) Nagoya Math. J. **11** (1957), 53 – 60.
- [29] G. D. Mostow, *Factor spaces of solvable groups*. Ann. of Math. (2) **60** (1954). 1 – 27.
- [30] G. D. Mostow, *On covariant fiberings of Klein spaces. I*. Amer. J. Math. **77** (1955) 247 – 278. *Covariant fiberings of Klein spaces. II*. Amer. J. Math. **84** 1962 466 – 474.
- [31] K. Oeljeklaus and W. Richthofer, *Recent results on homogeneous complex manifolds*. In: Complex analysis, III (College Park, Md., 1985-86), 78 – 119, Lecture Notes in Math., **1277**, Springer, Berlin, 1987.
- [32] K. Oeljeklaus and W. Richthofer, *On the structure of complex solvmanifolds*. J. Differential Geom. **27** (1988), no. 3, 399 – 421.
- [33] M. S. Raghunathan, “Discrete subgroups of Lie groups”, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 68. Springer-Verlag, New York-Heidelberg, 1972.
- [34] J.-P. Serre, *Quelques problèmes globaux relatifs aux variétés de Stein*, In: Coll. sur les fonctions de plusieurs variables, Bruxelles (1953), 57 – 68.

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF REGINA, REGINA, CANADA S4S 0A2

E-mail address: `seyed.ruhala.ahmadi@gmail.com` and `gilligan@math.uregina.com`